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ADDENDUM

# Critical spectral dimensionalities of random walks and phase transitions on fractals

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**Abstract.** For random walks and phase transitions on fractals, there exist various critical spectral dimensionalities, which are direct generalizations of the corresponding critical space dimensions in the case of Euclidean spaces.

In a previous letter [1] we discussed random walks and self-avoiding walks on a class of self-affine fractals, the bifractals [2], whose spectral dimensions are often larger than 2, a critical value of spectral dimension of random walks on fractals. Here we add some discussion on the issue of critical dimensions about fractals.

It is well known that there are various critical space dimensions for random walks and phase transitions. For example, the lower critical dimension of a random walk is 2, i.e. a completely stochastic random walk can only be embedded in a space with dimension not less than 2. The upper critical dimension is 4, i.e. two random walks do not intersect in more than 4 dimensions. These conclusions concern translationally invariant systems. Now we treat the corresponding problem in scale-invariant systems—the fractals.

First we consider the intersection of two fractal subsets $A$ and $B$ on a fractal $F$ with its fractal dimensionality $d$. The fractal dimension of the intersection is [3]

$$d_{A\cap B} = d_A + d_B - d \tag{1}$$

where $d_A$ and $d_B$ are respectively the fractal dimensions of $A$ and $B$. Since $A \subseteq F$,

$$d_A \leqslant d. \tag{2}$$

Let $A$ and $B$ belong to the same class of fractals, then

$$d_A = d_B. \tag{3}$$

That $A$ and $B$ intersect means that

$$d_{A\cap B} \geqslant 0. \tag{4}$$

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Thus by equation (1) we obtain

$$d \leqslant 2d_A. \tag{5}$$

Our present problem is random walks on fractals, i.e. $A$ is a random walk, $d_A = d_w$, the random walk dimension $d_w$ is related to the fractal dimension $d$ and the spectral dimensionality $\tilde{d}$ of the fractal substrate by $\tilde{d} = 2d/d_w$ [4]. Thus by equations (2) and (5), we know that 2 and 4 are respectively the lower and upper critical spectral dimensionalities.

The fractal dimensionalities of fractals usually studied are less than 2, the lower critical spectral dimensionality. We have shown that for a class of self-affine fractals, referred to as 'bifractals', the spectral dimensionalities are not less than 2 [1, 2]. On bifractals, it can easily be shown that the critical spectral dimensionalities remain the same as above [5].

It has been shown that the critical space dimensions of phase transitions are also generalized to the critical spectral dimensionalities in the case of fractals, with the well known values unchanged. In most cases, a continuous symmetry cannot be spontaneously broken if $\tilde{d} \leqslant 2$ [6, 7], while a discreet symmetry can be spontaneously broken only if $\tilde{d} \geqslant 2$ [7]. In these proofs, random walks are related to phase transitions, indicating that the equality of their critical spectral dimensions (or critical space dimensions in the case of Euclidean spaces) is due to their internal connections.

One may also convince himself that the upper critical spectral dimensionality is also that of phase transitions, i.e. fluctuations are negligible on fractals whose spectral dimensionalities are larger than 4, thus the Ginzburg criterion [8] is generalized to fractals. This result could be obtained by reasonably adopting the results of the correlation function for a certain mode and the mean value of the order parameter for translationally invariant systems. In a similar way to that for translationally invariant systems [8], one may estimate that the fluctuation over the correlation length is $\Gamma(\xi) \sim \xi^{2-\tilde{d}}$ and that the mean value of the order parameter $\langle m \rangle \propto \xi^{-1}$. Therefore $\Gamma(\xi) \ll \langle m \rangle^2$ can be fulfilled in the limit $\xi \to \infty$ only if $\tilde{d} > 4$.

To conclude, various critical space dimensions of random walks and phase transitions in the case of Euclidean spaces are generalized to the corresponding critical spectral dimensions in the more general case of fractals or even disordered systems with 'abnormal' spectral dimensions different from the space dimensions, making the former a special case of the latter. This reminds us that, although they reduce to the same in translationally invariant systems, the 'dimensional effect' in physics may be geometric, topological or spectral.

## References

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